Consider the open half space $\mathbb{H} := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$.

Let $\boldsymbol{\gamma}\colon\ [a,b] \to \mathbb{H}$ be a C^1-curve. $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$

Regular: The velocity vector is nowhere zero

Definition 8.18 The **hyperbolic length** of $\boldsymbol{\gamma}$ is

$$\mathcal{L}_{\mathrm{h}}(\boldsymbol{\gamma}) := \int_a^b \frac{|\dot{\boldsymbol{\gamma}}(t)|}{\gamma_n(t)}\,\mathrm{d}t.$$

Definition 8.19 A **hyperbolic geodesic** is a curve $\boldsymbol{\gamma}\colon\ I \to \mathbb{H}$, defined on some interval $I \subset \mathbb{R}$, with the following properties.

(i) The parametrisation is proportional to arc length, i.e. the function $|\dot{\boldsymbol{\gamma}}|/\gamma_n$ is constant.

(ii) The curve is locally distance minimising, i.e. for sufficiently small subintervals $[t_0, t_1] \subset I$, the curve $\boldsymbol{\gamma}|_{[t_0,t_1]}$ is the shortest connection from $\boldsymbol{\gamma}(t_0)$ to $\boldsymbol{\gamma}(t_1)$.[7]

Example 8.20

Consider two points: $(a_1, \ldots, a_{n-1}, e^a)$ and $(a_1, \ldots, a_{n-1}, e^b)$ $(b > a)$

The unique hyperbolic geodesic is $\vec{\gamma}(s) = (a_1, \ldots, a_{n-1}, e^s)$, $s \in [a, b]$

(1) $\dfrac{|\vec{\gamma}\,'(s)|}{\gamma_n(s)} = \dfrac{e^s}{e^s} = 1$, hence $L_h(\gamma) = b - a$

(2) Let $\vec{\beta} : [0,1] \to \mathbb{H}$ such that $\vec{\beta}(0) = \vec{\gamma}(a)$, $\vec{\beta}(1) = \vec{\gamma}(b)$

Then we have $\dfrac{|\dot{\vec{\beta}}(t)|}{\beta_n(t)} \geq \dfrac{\dot{\beta}_n(t)}{\beta_n(t)}$

$$L_h(\vec{\beta}) = \int_0^1 \frac{|\dot{\vec{\beta}}(t)|}{\beta_n(t)} dt \geq \int_0^1 \frac{\dot{\beta}_n(t)}{\beta_n(t)} dt = \log \beta_n(t) \Big|_0^1 = \log e^b - \log e^a = b - a$$

Equality if and only if $\dot{\beta}_1 = \cdots = \dot{\beta}_{n-1} = 0$ and $\dot{\beta}_n > 0$ (regular)

Definition 8.21 The **tangent space** $T_{\mathbf{x}}\mathbb{H}$ of $\mathbb{H}$ at a point $\mathbf{x} \in \mathbb{H}$ is the space of velocity vectors of C^1-curves passing through $\mathbf{x}$:

$$T_{\mathbf{x}}\mathbb{H} := \{\dot{\boldsymbol{\gamma}}(0), \text{ where } \boldsymbol{\gamma}\colon (-\varepsilon, \varepsilon) \to \mathbb{H} \text{ is a } C^1\text{-curve with } \boldsymbol{\gamma}(0) = \mathbf{x}\}.$$

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathrm{h}} := \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{x_n^2} \qquad |\mathbf{v}|_{\mathrm{h}} := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_{\mathrm{h}}} = \frac{|\mathbf{v}|}{x_n} \longrightarrow \mathcal{L}_{\mathrm{h}}(\boldsymbol{\gamma}) = \int_a^b |\dot{\boldsymbol{\gamma}}(t)|_{\mathrm{h}}\, \mathrm{d}t$$

Riemannian metric the half-space model of hyperbolic space

Definition 8.23 Given a C^1-map $\Phi\colon \mathbb{H} \to \mathbb{H}$, its **differential** at the point $\mathbf{x} \in \mathbb{H}$ is the linear map

$$\begin{array}{ccc} T_{\mathbf{x}}\mathbb{H} & \longrightarrow & T_{\Phi(\mathbf{x})}\mathbb{H} \\ \mathbf{v} & \longmapsto & J_{\Phi,\mathbf{x}}(\mathbf{v}) \end{array}$$

If $\vec{x} = \vec{\gamma}(0)$,

$$\frac{d}{dt}(\Phi \circ \vec{\gamma})(0) = J_{\Phi,\vec{x}}(\dot{\vec{\gamma}}(0))$$

↳ velocity vector of $\vec{\gamma}$ at $\vec{x}$ maps to velocity vector of $\Phi \circ \gamma$ at $\Phi(\vec{x})$

Definition 8.24 An **isometry** of $\mathbb{H}$ is a diffeomorphism $\Phi\colon \mathbb{H} \to \mathbb{H}$ whose differential preserves the Riemannian metric, that is,

$$\underbrace{\langle J_{\Phi,\mathbf{x}}(\mathbf{v}), J_{\Phi,\mathbf{x}}(\mathbf{w})\rangle_{\mathrm{h}}}_{\Phi(\vec{x})} = \underbrace{\langle \mathbf{v}, \mathbf{w}\rangle_{\mathrm{h}}}_{\vec{x}}$$

for all $\mathbf{x} \in \mathbb{H}$ and all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}\mathbb{H}$.

Definition 8.25 The **angle** $\measuredangle(\mathbf{v}, \mathbf{w}) \in [0, \pi]$ between two tangent vectors $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}\mathbb{H}$ is determined by

$$\cos \measuredangle(\mathbf{v}, \mathbf{w}) := \frac{\langle \mathbf{v}, \mathbf{w}\rangle_{\mathrm{h}}}{|\mathbf{v}|_{\mathrm{h}} \cdot |\mathbf{w}|_{\mathrm{h}}}.$$

$$= \frac{\dfrac{\langle \vec{v}, \vec{w}\rangle}{x_n^2}}{\dfrac{|\vec{v}|}{x_n} \cdot \dfrac{|\vec{w}|}{x_n}}$$

$$= \frac{\langle \vec{v}, \vec{w}\rangle}{|\vec{v}| \cdot |\vec{w}|} \longrightarrow \text{conformally equivalent}$$

$$\partial\mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n\colon\ x_n = 0\} \cup \underbrace{\{\infty\}}_{8.20} \qquad \text{Boundary at infinity}$$

Lemma 8.27 *Let* **p** $\neq \infty$ *be a point in* $\partial\mathbb{H}$. *The inversion* Φ *of* $\mathbb{R}^n \cup \{\infty\}$ *in a sphere centred at* **p** *restricts to an isometry of* $\mathbb{H}$.

$$\Phi(\vec{x}) = \vec{p} + r_0^2 \cdot \frac{\vec{x} - \vec{p}}{|\vec{x} - \vec{p}|^2}$$

(1) $\Phi : \mathbb{H} \to \mathbb{H}$

(2) Metric properties:

WLOG, $\vec{p} = \vec{0}$

Let $\vec{x}(t)$ be a C^1 curve in $\mathbb{H}$, $\vec{y} = r^2 \frac{\vec{x}}{|\vec{x}|^2}$ be its image curve

Goal: $|\dot{\vec{y}}|_h = |\dot{\vec{x}}|_h$

$$\dot{\vec{y}} = r^2 \frac{\dot{\vec{x}}}{|\vec{x}|^2} - r^2 \vec{x} \frac{2\langle \dot{\vec{x}}, \vec{x} \rangle}{|\vec{x}|^4}$$

$$|\dot{\vec{y}}| = r^2 \frac{|\dot{\vec{x}}|}{|\vec{x}|^2}$$

$$\frac{|\dot{\vec{y}}|}{|\vec{y}|} = r^2 \frac{|\dot{\vec{x}}|}{|\vec{x}|^2} \cdot \frac{|\vec{x}|}{r^2} = \frac{|\dot{\vec{x}}|}{|\vec{x}|}.$$

Since $\vec{y} = k\vec{x}$, $\frac{|\vec{x}|}{|\vec{y}|} = \frac{x_n}{y_n}$

$$|\dot{\vec{y}}|_h = \frac{|\dot{\vec{y}}|}{y_n} = \frac{|\dot{\vec{x}}|}{x_n} = |\dot{\vec{x}}|_h$$

Definition 8.28 A **hyperbolic line** is a maximal geodesic (understood to be of unit speed) in $\mathbb{H}$, i.e. a geodesic $\gamma\colon \mathbb{R} \to \mathbb{H}$ defined on all of $\mathbb{R}$.

Proposition 8.29 *The hyperbolic lines are precisely the euclidean half-lines orthogonal to $\partial\mathbb{H}$ and the euclidean semicircles orthogonal to $\partial\mathbb{H}$.*

(1) euclidean half-line $s \mapsto (a_1, \dots, a_{n-1}, e^s)$, $s \in \mathbb{R}$

(2)

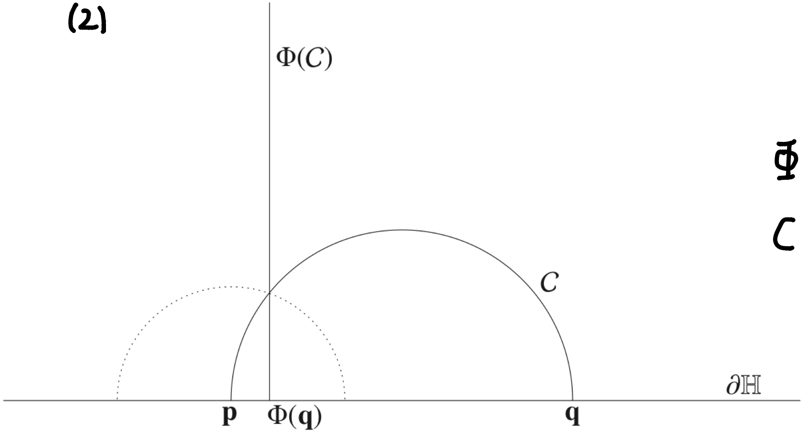


Figure 8.10 Hyperbolic lines.

Φ is isometry of $\mathbb{H}$ (lemma 8.27)

$C \xrightarrow[\Phi]{} $ half-line (Prop. 8.10)

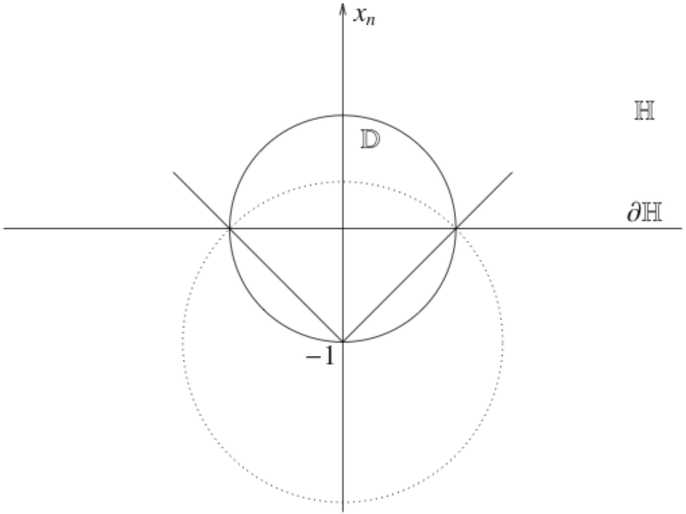


Figure 8.12 The inversion sending $\mathbb{H}$ to $\mathbb{D}$.

inversion of the sphere of radius $\sqrt{2}$ about the point $(0,\ldots,0,-1)$

Proposition 8.30 *The Riemannian metric on $\mathbb{D}$ induced from the hyperbolic metric on $\mathbb{H}$ via the diffeomorphism $\Phi\colon \mathbb{H} \to \mathbb{D}$ is given by*

$$\boxed{\langle \mathbf{v}, \mathbf{w} \rangle_{h'} = \frac{4\langle \mathbf{v}, \mathbf{w} \rangle}{(1 - |\mathbf{y}|^2)^2} \quad \textit{for} \;\; \mathbf{v}, \mathbf{w} \in T_{\mathbf{y}}\mathbb{D}.}$$

Let $\vec{x}(t)$ be a C^1 curve in $\mathbb{H}$, $\vec{y}(t) = \Phi(\vec{x}(t))$ in $\mathbb{D}$

Write $\vec{u} := (0, \dots, 0, 1)$. Then $\vec{x} + \vec{u} = \dfrac{2(\vec{y} + \vec{u})}{|\vec{y} + \vec{u}|^2}$

Goal : $|\dot{\vec{y}}|_{h'} = |\dot{\vec{x}}|_h$

$$|\dot{\vec{x}}|_h = \frac{|\dot{\vec{x}}|}{x_n} = \frac{|\dot{\vec{x}}|}{\langle \vec{x}, \vec{u} \rangle}$$

$$\dot{\vec{x}} = \frac{2\dot{\vec{y}}\,|\vec{y} + \vec{u}|^2 - 2(\vec{y} + \vec{u}) \cdot 2\langle \dot{\vec{y}}, \vec{y} + \vec{u} \rangle}{|\vec{y} + \vec{u}|^4} \longrightarrow |\dot{\vec{x}}| = \frac{2|\dot{\vec{y}}|}{|\vec{y} + \vec{u}|^2}$$

$$\langle \vec{x}, \vec{u} \rangle = \frac{2(\langle \vec{y}, \vec{u} \rangle + 1)}{|\vec{y} + \vec{u}|^2} - 1 = \frac{|\vec{y} + \vec{u}|^2 - |\vec{y}|^2 - |\vec{u}|^2 + 1}{|\vec{y} + \vec{u}|^2} - 1 = \frac{1 - |\vec{y}|^2}{|\vec{y} + \vec{u}|^2}$$

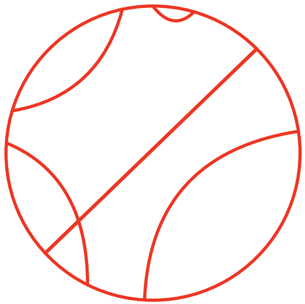
$$|\dot{\vec{y}}|_{h'} = |\dot{\vec{x}}|_h = \frac{2|\dot{\vec{y}}|}{|\vec{y} + \vec{u}|^2} \cdot \frac{|\vec{y} + \vec{u}|^2}{1 - |\vec{y}|^2} = \frac{2|\dot{\vec{y}}|}{1 - |\vec{y}|^2}$$

Definition 8.31 The pair $(\mathbb{D}, \langle\, .\,, .\, \rangle_{h'})$ is called **the Poincaré disc model of hyperbolic space**.

Proposition 8.32 *The hyperbolic lines in the Poincaré disc model are precisely the arcs of circles orthogonal to $\partial\mathbb{D}$ and the diameters of $\mathbb{D}$.*

$\Phi : \mathbb{H} \to \mathbb{D}$ is an isometry by the construction of metric

$$\begin{tikzcd}
\text{euclidean lines or circles orthogonal to } \partial\mathbb{H} \arrow[r, "\Phi\ (\text{Prop. } 8.10)"] \arrow[d, Leftrightarrow, "(\text{Prop. } 8.29)"'] & \text{lines and circles orthogonal to } \partial\mathbb{D} \arrow[d, Leftrightarrow] \\
\text{hyperbolic lines in } \mathbb{H} \arrow[r, "\Phi"] & \text{hyperbolic lines in } \mathbb{D}
\end{tikzcd}$$



Theorem 8.33 (Osipov, Belbruno) *The inversion in the unit sphere $S^2 \subset \mathbb{R}^3 \cup \{\infty\}$ yields a bijection between, on the one hand, the hodographs of Kepler solutions with energy $h = 1/2$ or $h = 0$ and, on the other hand, the oriented hyperbolic lines in $\mathbb{D}^3$ or euclidean lines in $\mathbb{R}^3$, respectively.*[11] $h = \frac{1}{2}v^2 - \frac{\mu}{r}$ (3.6)

radial plane $\longrightarrow$ $\mathbb{D}^2$ or $\mathbb{R}^2$

(Hyperbolic: $h = \frac{1}{2}$) For $c \neq 0$, hodograph is the part of velocity circle that outside the energy circle $\{v^2 = 1\}$

Inversion maps this to arc of circle orthogonal to $\partial\mathbb{D}^2$

For $c = 0$, hodograph is the part of radial ray outside energy circle

Inversion maps this to diameter of $\mathbb{D}^2$

↳ hyperbolic lines in $\mathbb{D}^2$ (Prop. 8.32)

(Parabolic: $h = 0$) For $c \neq 0$, hodographs are circles through the origin with origin removed $\{v^2 = 0\}$

For $c = 0$, hodographs are radial lines without origin, but include ∞

Inversion maps these to euclidean lines in $\mathbb{R}^2$

Proposition 8.34 *Let $s \mapsto \mathbf{v}(s)$ be the velocity curve of a Kepler solution with $h = 0$. The inversion in the unit sphere sends this curve to a euclidean line parametrised by arc length with respect to the scaled euclidean metric $4\langle.,.\rangle$.*

Parabolic

xy-plane, $\vec{c} = (0,0,c)$, $\vec{e} = (1,0,0)$

For $c \neq 0$, $\vec{v}(\theta) = \frac{\mu}{c}(-\sin\theta, 1+\cos\theta)$, $\theta \in (-\pi,\pi)$ (8.1)

$\hookrightarrow v^2 = \frac{2\mu^2}{c^2}(1+\cos\theta)$

Let $\vec{w}$ be the curve by inverting $\vec{v}$ in the unit circle

$$\vec{w}(\theta) = \frac{\vec{v}(\theta)}{v^2(\theta)} = \frac{c}{2\mu}\left(\frac{-\sin\theta}{1+\cos\theta}, 1\right)$$

$\vec{r} = (r\cos\theta, r\sin\theta) = \left(\frac{1}{2}\left(\frac{c^2}{\mu} - u^2\right), \frac{c}{\sqrt{\mu}}u\right)$ (Section 4.3)

$$\frac{c}{2\mu} \cdot \frac{-\sin\theta}{1+\cos\theta} = \frac{-\sin\theta}{2c} \cdot \underbrace{\boxed{\frac{\frac{c^2}{\mu}}{1+\cos\theta}}}_{=r\text{, see Ch.3}} = \frac{-r\sin\theta}{2c} = -\frac{1}{2}\cdot\frac{u}{\sqrt{\mu}} = -\frac{s}{2}$$

$\hookrightarrow \vec{w}(s) = \left(-\frac{s}{2}, \frac{c}{2\mu}\right)$ $\quad \left|\frac{dw}{ds}\right|^2_* = 1 \Rightarrow \langle\cdot,\cdot\rangle_* = 4\langle\cdot,\cdot\rangle$

For $c \to 0$,

$\vec{r} = \left(-\frac{u^2}{2}, 0\right)$

$\vec{v} = (\dot{x}, 0) = (-u\dot{u}, 0) = \left(-\frac{2}{s}, 0\right)$

$\hookrightarrow \frac{u^3}{6} = \sqrt{\mu}\,t$ (Prop. 4.7) $\Rightarrow \frac{u^2\dot{u}}{2} = \sqrt{\mu}$

$\vec{w} = \left(\frac{1}{\dot{x}}, 0\right) = \left(-\frac{s}{2}, 0\right)$

Horizontal lines in upper half oriented in negative x-direction

Hyperbolic **Proposition 8.36** *Let $u \mapsto \mathbf{v}(u)$ be the velocity curve of a Kepler solution with $h = 1/2$. The inversion in the unit sphere sends this curve to a hyperbolic line parametrised by arc length.*

xy-plane, $\vec{c} = (0,0,c)$, $\vec{e} = (e,0,0)$, $e > 1$

For $c \neq 0$,

$\vec{v}(\theta) = \frac{1}{\sqrt{e^2-1}}(-\sin\theta, e+\cos\theta)$, $1 + e\cos\theta > 0$. (8.1) $c = \mu\sqrt{e^2-1}$ (3.7)

$\hookrightarrow v^2 = \frac{1+e^2+2e\cos\theta}{e^2-1}$

$\vec{w} = \frac{\vec{v}}{v^2} = \frac{\sqrt{e^2-1}}{1+e^2+2e\cos\theta}(-\sin\theta, e+\cos\theta)$

$\left|\frac{d\vec{w}}{du}\right|_{h'} = \frac{2\boxed{\left|\frac{d\vec{w}}{du}\right|}}{1-w^2}$ $\rightarrow \left|\frac{d\vec{w}}{d\theta}\right| \cdot \left|\frac{d\theta}{dt}\right| \cdot \left|\frac{dt}{du}\right|$

$\left|\frac{d\vec{w}}{d\theta}\right| = \frac{\sqrt{e^2-1}}{1+e^2+2e\cos\theta}$, $\left|\frac{d\theta}{dt}\right| = \frac{c}{r^2}$, $\left|\frac{dt}{du}\right| = r \rightarrow \dot{u} = \sqrt{\frac{\mu}{a}} \cdot \frac{1}{r}$ (4.2), $a = \frac{\mu}{2h} = \mu$ (3.8)

$1 - w^2 = \frac{2(1+e\cos\theta)}{1+e^2+2e\cos\theta} = \frac{\frac{2c^2}{\mu}}{r(1+e^2+2e\cos\theta)}$

$\left|\frac{d\vec{w}}{du}\right|_{h'} = \sqrt{e^2-1} \cdot \frac{\mu}{c} = 1$

For $c=0,\ e=1$,

$x(u)=a(1-\cosh u)$, $t(u)=a(\sinh u-u)$ (see Ex. 4.3)

$$\dot{x}=\frac{dx}{du}\cdot\frac{du}{dt}=\frac{dx}{du}\cdot\left(\frac{dt}{du}\right)^{-1}=\frac{-\sinh u}{\cosh u-1}$$

$$\hookrightarrow \vec{w}=\left(\frac{1}{\dot{x}},0\right)=\left(\frac{1-\cosh u}{\sinh u},\ 0\right)$$

$$\left|\frac{d\vec{w}}{du}\right|=\left|\frac{1-\cosh u}{\sinh^2 u}\right|=\frac{\cosh u-1}{\sinh^2 u}$$

$$1-w^2=\frac{2(\cosh u-1)}{\sinh^2 u}$$

$$\left|\frac{d\vec{w}}{du}\right|_{h'}=\frac{2\left|\frac{dw}{du}\right|}{1-w^2}=1$$

For $K \in \mathbb{R}^+$ consider the sphere $S^3_{1/\sqrt{K}} \subset \mathbb{R}^4$ of radius $1/\sqrt{K}$ centred at $\mathbf{0}$. The stereographic projection $\Psi_K\colon\ S^3_{1/\sqrt{K}} \setminus \{N\} \to \mathbb{R}^3$, where $N = (0, 0, 0, 1/\sqrt{K})$, is given by

$$\Psi_K(x_1, x_2, x_3, x_4) = \frac{1}{1 - \sqrt{K}\,x_4}(x_1, x_2, x_3);$$

the inverse map is given by

$$\Psi_K^{-1}(w_1, w_2, w_3) = \frac{1}{Kw^2 + 1}\left(2w_1, 2w_2, 2w_3, \frac{Kw^2 - 1}{\sqrt{K}}\right),$$

where we write $\mathbf{w} = (w_1, w_2, w_3)$ and $w = |\mathbf{w}|$.

Proposition 8.38 *The Riemannian metric $\langle\,.\,,.\,\rangle_K$ on $\mathbb{R}^3$ that turns Ψ_K into an isometry is given by*

$$\boxed{\langle\,.\,,.\,\rangle_K = \frac{4\langle\,.\,,.\,\rangle}{(1 + Kw^2)^2}.} \tag{8.12}$$

$$|\dot{\vec{w}}|_K^2 = |\dot{\vec{x}}|^2 = \frac{4|\dot{\vec{w}}|^2}{(1+Kw^2)^2} \qquad \vec{x} = \Psi_K^{-1}(\vec{w})$$

For $K \in \mathbb{R}^-$ we can define a Riemannian metric on the disc $\mathbb{D}_{1/\sqrt{|K|}}$ of radius $1/\sqrt{|K|}$ by the same formula (8.12). For $K = 0$ this formula defines the scaled euclidean metric $4\langle\,.\,,.\,\rangle$ on $\mathbb{R}^3$.

Definition 8.39 The manifolds

$$M_K := \begin{cases} \mathbb{D}_{1/\sqrt{|K|}} & \text{for } K < 0 \\ \mathbb{R}^3 & \text{for } K = 0 \\ S^3_{1/\sqrt{K}} & \text{for } K > 0 \end{cases}$$

with the Riemannian metric $\langle . , . \rangle_K$ are called the three-dimensional **space forms**.

Definition 8.40 Let $\mathbf{r}$ be a solution of the Kepler problem with $t = 0$ the time of pericentre passage or of (regularised) collision with the force centre, respectively. The function

$$\boxed{s(t) := \int_0^t \frac{\mathrm{d}\tau}{r(\tau)}}$$ (Exercise 4.6)

is called the **Levi-Civita parameter**.

Theorem 8.42 (Moser, Osipov, Belbruno) *The inversion of $\mathbb{R}^3$ in the unit sphere S^2 gives a one-to-one correspondence between the velocity curves of Kepler solutions with energy h, parametrised by the Levi-Civita parameter, and the unit speed geodesics in the space form M_{-2h}.*

1. One-to-one correspondance

a) $h > 0$, $K < 0 \rightarrow M_K = \mathbb{D}_{1/\sqrt{|K|}}$

inversion in S^2 maps $\{v^2 = 2h\}$ to $\{w^2 = \frac{1}{2h}\} = \partial \mathbb{D}_{1/\sqrt{2h}}$

b) $h < 0$, $K > 0 \rightarrow M_K = S^3_{1/\sqrt{K}}$

Ψ_K maps great circles on $S^3_{1/\sqrt{K}}$ to circles

invariant under the negative inversion $\vec{w} \mapsto -\frac{\vec{w}}{Kw^2}$ in sphere of radius $1/\sqrt{K}$

$\langle \vec{w}_1, \vec{w}_2 \rangle = -\frac{1}{K}$, $\vec{w}_1, \vec{w}_2$ are intersection points of circle and radial line

inversion in $S^2 \rightarrow$ circles that $\langle \vec{v}_1, \vec{v}_2 \rangle = -K = 2h$ (lemma 8.6)

2. Metric parametrisation

(K) can be rewritten as $\dot{\vec{v}} = -\frac{\mu}{r^2} \cdot \frac{\vec{r}}{r}$

↳ $|\dot{\vec{v}}| = \frac{\mu}{r^2}$

$$\left|\frac{d\vec{v}}{ds}\right|_* = 1$$

Propose $\left|\frac{d\vec{v}}{ds}\right|_* = \frac{4\left|\frac{d\vec{v}}{ds}\right|}{(v^2-2h)^2}$

$$\left|\frac{d\vec{v}}{ds}\right| = |\dot{\vec{v}}| \cdot \frac{dt}{ds} = \frac{\mu}{r^2} \cdot r = \frac{1}{2}v^2 - h \quad (3.6)$$

Polarise we get the metric $\frac{4\langle \cdot , \cdot \rangle}{(v^2-2h)^2}$

↳ by inversion, $\frac{4\langle \cdot , \cdot \rangle}{(v^2-2h)^2} = \frac{4\langle \cdot , \cdot \rangle}{(1-\underline{2h}w^2)^2} = \langle \cdot , \cdot \rangle_{-2h}$